

3D Computer Vision Camera Models

Nassir Navab

based on a course given at UNC by Marc Pollefeys & the book "Multiple View Geometry" by Hartley & Zisserman

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chair for computer aided medical procedures

department of computer science | technische universität münchen



Outline of the Computer Vision Lecture

Projective Geometry and Transformations

- Transformations with homogeneous coordinates
- 2D projective geometry
- 3D projective geometry
- 1D projective geometry

Estimation

- Direct and Iterative Estimation
- DLT Algorithm
- Cost Functions
- Robust Estimation (RANSAC)

Algorithm Evaluation and Error Analysis

- Statistics, Error Propagation

Camera Geometry

- Camera Models
- Camera Calibration

Advanced Estimation

- Bundle Adjustment
- Iterative Closest Point (ICP)

Single-View Geometry

Two-View Geometry

- Epipolar Geometry / Fundamental

- 3D Reconstruction of Cameras and

Matrix / Essential Matrix

Structure

Three-View Geometry

- Trifocal Tensor
- Computation of the Trifocal Tensor

N-View Geometry

- N-Linearities and Multiple View Tensors
- N-View Computational Methods
- Factorization
- Structure From Motion



Outline – Camera Model

- History of Camera (Camera Obscura)
- Intensity Images
- Geometric Parameters of a Finite camera
- Projective Camera
 - Camera Center, Principal Plane, Axis Plane, Principal Point, Principal Ray
 - Decomposition of Camera Matrix
- Cameras at Infinity
- Other Camera Models (Pushbroom and Line Cameras)



History :: Camara Obscura

http://en.wikipedia.org/wiki/Camera_obscura



- **Camera obscura** (||Camera obscura) [LL. camera chamber + L. obscurus, obscura, dark.] (Opt.)
- 1. An apparatus in which the images of external objects, formed by a convex lens or a concave mirror, are thrown on a paper or other white surface placed in the focus of the lens or mirror within a darkened chamber, or box, so that the outlines may be traced.
- (Photog.) An apparatus in which the image of an external object or objects is, by means of lenses, thrown upon a sensitized plate or surface placed at the back of an extensible darkened box or chamber variously modified; commonly called simply the camera.

Websters Dictionary, 1913

illum in tabula per radios Solis, quam in cœlo contingit: hoc eft,fi in cœlo fuperior pars deliquiñ patiatur,in radiis apparebit inferior deficere,vt ratio exigit optica.



Sic nos exacté Anno .1544. Louanii eclipfum Solis obferuauimus, inuenimusq; deficere paulò plus g dex-

Camera Obscura, Reinerus Gemma-Frisius, 1544, published in "*De Radio Astronomica et Geometrica*", 1545 from Gernsheim, H., "The Origins of Photography"



History :: Camera Obscura





Camera Obscura, Athanasius Kircher, 1646 from Gernsheim, H., *"The Origins of Photography"*

Portable 'Tent' Camera Obscura, Johannes Kepler, 1620 from Gernsheim, H., "The Origins of Photography"



History :: Camera Obscura



Reflex Camera Obscura, Johannes Zahn, 1685 published in Zahn, J., "*Oculus Artificialis*", (1685 - 1686) from Gernsheim, H., "The Origins of Photography"



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Intensity Image

- photometric parameters
 - light energy reaching the sensors after being reflected from objects (not discussed in this course)

geometric parameters

 2D position in the image on which a 3D point is projected (depending on camera geometry, discussed here)



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Pinhole Camera Model





mapping between 3D world and 2D image

$$(X, Y, Z)^{T} \mapsto (x, y)^{T}$$

- central projection
- models are described by matrices with particular properties



Homogeneous Coordinates





$$(X, Y, Z, 1)^{\mathrm{T}} \mapsto (x, y, z)^{\mathrm{T}}$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & 0 \\ f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Central Projection







$$(X, Y, Z)^{T} \mapsto (fX/Z + p_x, fY/Z + p_y)^{T}$$

where
$$(p_x, p_y)^T$$

are the coordinates of the principal point

Principal Point Offset

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{f} & \mathbf{p}_{\mathbf{x}} \\ & \mathbf{f} & \mathbf{p}_{\mathbf{y}} \\ & & 1 \end{bmatrix}$$

is called camera calibration matrix















Camera Rotation and Translation





CCD Cameras :: Non-Square Pixel









Skew Parameter

S skew parameter

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

5 DOF

 $P = KR \begin{bmatrix} I & | & -\widetilde{C} \end{bmatrix}$ finite projective camera with 11 DOF



 $\mathbf{x} = P\mathbf{X}$

$$P = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix} \begin{bmatrix} R & -R\widetilde{C} \\ 0 & 1 \end{bmatrix}$$
$$= KR \begin{bmatrix} I \mid -\widetilde{C} \end{bmatrix}$$

projection matrix 11 DOF (5+3+3)

non-singular



non-singular 3x3 matrix (8 DOF)

$$M = KR$$
$$P = KR[I | -\widetilde{C}]$$
$$P = M[I | -\widetilde{C}]$$

decompose projection matrix P in K,R,C

$$P = \begin{bmatrix} M | \mathbf{p}_4 \end{bmatrix} \qquad \begin{bmatrix} K, R \end{bmatrix} = \underbrace{RQ}_{\checkmark}(M) \qquad \widetilde{C} = -M^{-1}\mathbf{p}_4$$

RQ matrix
decomposition



- camera matrix P are identical with the set of homogeneous 3x4 matrix for which the left hand 3x3 submatrix is non singular
- {finite cameras}={P| det M \neq 0} ={P| rank(M)=3}
- If rank(P)=3, but rank(M)<3, then camera at infinity</p>
- if rank(P)<3 the matrix mapping will be a line or a point and not a plane (not a 2D image)



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Camera Anatomy

- camera center
- column vectors
- principal plane
- axis plane
- principal point
- principal ray





$$PC = 0$$
 P has a 1D null-space

we will proof that the 4-vector C is the camera center

$$\begin{split} X(\lambda) &= \lambda A + (1 - \lambda)C \quad \text{points on a line through A and C} \\ x &= PX = \lambda PA + (1 - \lambda)PC = \lambda PA \quad \text{since} \quad PC = 0 \end{split}$$

all 3D points on the line are mapped on the same 2D image point, and thus the line is a ray through the camera center

Finite cameras:

$$C = \begin{pmatrix} \widetilde{C} \\ 1 \end{pmatrix} = \begin{pmatrix} -M^{-1}\mathbf{p}_{4} \\ 1 \end{pmatrix}$$
Infinite cameras:

$$C = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix}, M\mathbf{d} = 0$$



Column Vectors





column vectors are the image points, which project the axis directions (X,Y,Z) and the origin

example for the image of the y-axis (vanishing pt) $[\mathbf{p}_2] = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3 \ \mathbf{p}_4 [0 \ 1 \ 0 \ 0]^T$

$$\mathbf{p}_4$$
 is the image of the world origin



Row Vectors

$$\mathbf{p}^i \in \mathbb{R}^4, i = 1,...,3$$

represent geometrically particular world planes

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1T} \\ \mathbf{p}^{2T} \\ \mathbf{p}^{3T} \end{bmatrix} \text{ row vectors}$$
$$= \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \end{bmatrix} \text{ column vectors}$$

Row Vectors of the Projection Matrix

 P^1 is defined by the camera center and the line x=0 on the image

 P^2 is defined by the camera center and the line y=0 on the image

Example P²

$$\begin{bmatrix} x \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1^{\mathsf{T}}} \\ \mathbf{p}^{2^{\mathsf{T}}} \\ \mathbf{p}^{3^{\mathsf{T}}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$



$$\mathbf{p}^{2^{\mathrm{T}}}\mathbf{X} = \mathbf{0}$$
$$\mathbf{p}^{2^{\mathrm{T}}}C = \mathbf{0}$$

respectively for P¹



plane through camera center and parallel to the image plane

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} p^{1^{\mathsf{T}}} \\ p^{2^{\mathsf{T}}} \\ p^{3^{\mathsf{T}}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

points **X** are imaged on the line at infinity

P³

principal plane

if **X** is on the principle plane

especially

$$\mathbf{p}^{3T}\mathbf{X} = \mathbf{0}$$
$$\mathbf{p}^{3T}\mathbf{C} = \mathbf{0}$$

Principle Point

line through camera center and perpendicular to principle plane is the principle axis

intersection of the principle axis with the image plane is the principle point

$$\hat{\mathbf{P}}^{3} = \begin{pmatrix} p_{31} & p_{32} & p_{33} & 0 \end{pmatrix}^{\mathrm{T}} \text{ normal direction to principal plane}$$
principle point
$$\mathbf{X}_{0} = \mathbf{P}\hat{\mathbf{P}}^{3} = \mathbf{M}\mathbf{m}^{3} \quad \text{where} \quad \mathbf{P} = \begin{bmatrix} \mathbf{M} \mid \mathbf{p}_{4} \end{bmatrix}$$
and
$$\mathbf{m}^{3^{\mathrm{T}}} \text{ third row of M}$$



ambiguity that principle axis points towards the front of the camera (positive direction)

$$\mathbf{x} = \mathbf{P}_{cam} \mathbf{X}_{cam} = \mathbf{K} \left[\mathbf{I} \mid \mathbf{0} \right] \mathbf{X}_{cam}$$

itive
$$c$$
 P_3 m^3 or $-m^3$
 P_2 P_1 P_2 P_2 P_1 P_2 $P_$

towards the front of the camera

direction unaffected by scaling

$$\mathbf{P} = k\mathbf{K}\mathbf{R}\left[\mathbf{I} \mid -\widetilde{\mathbf{C}}\right] = \left[\mathbf{M} \mid \mathbf{p}_{4}\right]$$

 $P_{cam} \mapsto kP_{cam} \quad v \mapsto k^4 v$

since det(R) > 0 $v = det(M)m^{3}$



Forward Projection

maps a point in space on the image plane $\mathbb{P}^3 \rightarrow \mathbb{P}^2$

 $\mathbf{x} = \mathbf{P}\mathbf{X}$

Points at infinity

$$D = (d^T, 0)^T$$

$$\mathbf{x} = \mathbf{P}\mathbf{D} = \left[\mathbf{M} \mid \mathbf{p}_4\right]\mathbf{D} = \mathbf{M}\mathbf{d}$$

only M effects the projection of points at infinity (vanishing points)



Back-projection to Rays

points on the reconstructed ray

PC = 0 camera center C

$$X = P^{+}X$$
 $P^{+} = P^{T}(PP^{T})^{-1}$ $PP^{+} = I$
(pseudo-inverse)

ray is the line formed by those two points

$$X(\lambda) = P^{+}x + \lambda C$$

$$D = \left((M^{-1}x)^{T}, 0 \right)^{T} \text{ intersection of the ray with the plane at infinity}$$

$$X(\mu) = \mu \left(\frac{M^{-1}x}{0} \right) + \left(\frac{-M^{-1}p_{4}}{1} \right) = \left(\frac{M^{-1}(\mu x - p_{4})}{1} \right)$$



х

Depth of Points

$$X = (X, Y, Z, 1)$$

$$C = (\widetilde{C}, 1)$$

$$w = P^{3^{T}}X = P^{3^{T}}(X - C) = m^{3^{T}}(\widetilde{X} - \widetilde{C})$$
(PC=0) (dot product)

If $\det M > 0$; $\|m^3\| = 1$, then m³ unit vector pointing in positive axis direction

Suppose $P(X, Y, Z, T)^{T} = w(x, y, 1)^{T}$. Then

depth(X;P) =
$$\frac{\text{sign}(\text{detM})w}{T \| \text{m}^3 \|}$$





depth(X;P) =
$$\frac{\text{sign}(\text{detM})w}{\|m^3\|}$$

$$P(X,Y,Z,1)^{\mathrm{T}} = w(x,y,1)^{\mathrm{T}}$$

 $\mathbf{x} = \mathbf{P}\mathbf{X} \qquad \mathbf{x} = \mathbf{P}\mathbf{X}$ $\mathbf{P} = \begin{bmatrix} \mathbf{I} \mid \mathbf{0} \end{bmatrix} \qquad \mathbf{P} = \mathbf{R}\begin{bmatrix} \mathbf{I} \mid -\widetilde{\mathbf{C}} \end{bmatrix}$ $depth(X; P) = Z \qquad depth(X; P) = \mathbf{R}^{3^{T}}(\widetilde{X} - \widetilde{\mathbf{C}})$



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Camera Matrix Decomposition

Finding the camera center C

PC = 0

numerically: find right null-space by SVD of P $P^{3x4} = U^{3x4}D^{4x4}V^{4x4^{T}} \rightarrow P^{3x4}V^{4x4} = U^{3x4}D^{4x4} \rightarrow PV^{4} = 0$

Algebraically:

$$X = \det([p_2, p_3, p_4]) \quad Y = -\det([p_1, p_3, p_4])$$
$$Z = \det([p_1, p_2, p_4]) \quad T = -\det([p_1, p_2, p_3])$$

where $\mathbf{C} = (X, Y, Z, T)^{\mathrm{T}}$



Camera Matrix Decomposition

Finding the camera center C

PC = 0

Any plane π going through C will be a linear combination of the three planes defined by the rows of P. Therefore:

$$det([\pi, P^{T}]) = 0$$

= $\pi_{1} det(P_{234}) - \pi_{2} det(P_{134}) + \pi_{3} det(P_{124}) - \pi_{4} det(P_{123})$
$$X = det([p_{2}, p_{3}, p_{4}]) \quad Y = -det([p_{1}, p_{3}, p_{4}])$$

$$Z = det([p_{1}, p_{2}, p_{4}]) \quad T = -det([p_{1}, p_{2}, p_{3}])$$

where $C = (X, Y, Z, T)^{T}$



Camera Matrix Decomposition

Finding the camera orientation and internal parameters

$$P = \left[M \mid -M\widetilde{C} \right] = K \left[R \mid -R\widetilde{C} \right] \qquad K = \begin{bmatrix} \alpha_x & s & y_0 \\ 0 & \alpha_y & x_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Decompose M = KR using RQ decomposition

$$=(\ Q \ R)^{-1} = R^{-1} \ Q^{-1}$$

Ambiguity removed by enforcing positive diagonal entries



$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix} \qquad \begin{array}{c} \operatorname{arctan(1/s)} \\ 1 & \gamma \end{array}$$

for CCD/CMOS, always s=0

Image from image, s≠0 possible (non coinciding principal axis)

resulting camera: HP where H is a 3x3 homography



general projective interpretation

$$P = \begin{bmatrix} 3 \times 3 \text{ homography} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \times 4 \text{ homography} \end{bmatrix}$$

Meaningful decomposition in K,R,t requires Euclidean image and space

Camera center is still valid in projective space

Principal plane requires affine image and space

Principal ray requires affine image and Euclidean space



Literature on Camera Models

- Chapter 6 in R. Hartley and A. Zisserman, "Multiple View Geometry", 2nd edition, Cambridge University Press, 2003.
- Chapter 3 in O. Faugeras, "Three-dimensional Computer Vision", MIT Press, 1993.
- Chapter 2 in E. Trucco and A. Verri, "Introductory Techniques for 3-D Computer Vision", Prentice Hall, 1998.
- H. Gernsheim, "The Origins of Photography", *Thames and Hudson*, 1982.
- A. Shashua. "Geometry and Photometry in 3D Visual Recognition", Ph.D. Thesis, MIT, Nov. 1992. AITR-1401.