# 3D Computer Vision Camera Models 

Nassir Navab<br>based on a course given at UNC by Marc Pollefeys \& the book "Multiple View Geometry" by Hartley \& Zisserman

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## Outline of the Computer Vision Lecture

Projective Geometry and
Transformations

## Single-View Geometry

## Two-View Geometry

- Epipolar Geometry / Fundamental Matrix / Essential Matrix
- 3D Reconstruction of Cameras and Structure


## Estimation

- Direct and Iterative Estimation
- DLT Algorithm
- Cost Functions
- Robust Estimation (RANSAC)


## Algorithm Evaluation and Error Analysis

- Statistics, Error Propagation



## Three-View Geometry

- Trifocal Tensor
- Computation of the Trifocal Tensor


## Camera Geometry

- Camera Models
- Camera Calibration


## Advanced Estimation

- Bundle Adjustment
- Iterative Closest Point (ICP)


## N-View Geometry

- N-Linearities and Multiple View Tensors
- N-View Computational Methods
- Factorization
- Structure From Motion


## Outline - Camera Model

- History of Camera (Camera Obscura)
- Intensity Images
- Geometric Parameters of a Finite camera
- Projective Camera
- Camera Center, Principal Plane, Axis Plane, Principal Point, Principal Ray
- Decomposition of Camera Matrix
- Cameras at Infinity
- Other Camera Models (Pushbroom and Line Cameras)


## History :: Camara Obscura

## http://en.wikipedia.org/wiki/Camera obscura

$\square \quad \square$

## History :: Camara Obscura

Camera obscura (||Camera obscura) [LL. camera chamber + L. obscurus, obscura, dark.] (Opt.)

1. An apparatus in which the images of external objects, formed by a convex lens or a concave mirror, are thrown on a paper or other white surface placed in the focus of the lens or mirror within a darkened chamber, or box, so that the outlines may be traced.
2. (Photog.) An apparatus in which the image of an external object or objects is, by means of lenses, thrown upon a sensitized plate or surface placed at the back of an extensible darkened box or chamber variously modified; commonly called simply the camera.
Websters Dictionary, 1913
illum in tabula per radios Solis, quaim in coelo contingit : hoc eft, fi in ccelo fuperior pars deliquiũ patiatur, in radiis apparebit inferior deficere, vt ratio exigit optica.


Sic nos exaCt À Anno.1544. Louanii celipfim Solis obferuauimus, inuenimusq́; deficere paulò plus $\not$ g̈ dex-
Camera Obscura, Reinerus Gemma-Frisius, 1544, published in "De Radio Astronomica et Geometrica", 1545 from Gernsheim, H., "The Origins of Photography"

## History :: Camera Obscura



Camera Obscura, Athanasius Kircher, 1646
from Gernsheim, H., "The Origins of Photography"

Portable 'Tent' Camera Obscura, Johannes Kepler, 1620
from Gernsheim, H., "The Origins of Photography"

## History :: Camera Obscura



Reflex Camera Obscura, Johannes Zahn, 1685 published in Zahn, J., "Oculus Artificialis", (1685-1686)
from Gernsheim, H., "The Origins of Photography"

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## Intensity Image

- photometric parameters
- light energy reaching the sensors after being reflected from objects (not discussed in this course)
- geometric parameters
- 2D position in the image on which a 3D point is projected (depending on camera geometry, discussed here)


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## Pinhole Camera Model




- mapping between 3D world and 2D image

$$
(\mathrm{X}, \mathrm{Y}, \mathrm{Z})^{\mathrm{T}} \mapsto(\mathrm{x}, \mathrm{y})^{\mathrm{T}}
$$

- central projection
- models are described by matrices with particular properties


## Homogeneous Coordinates




$$
(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, 1)^{\mathrm{T}} \mapsto(\mathrm{x}, \mathrm{y}, \mathrm{z})^{\mathrm{T}}
$$

$$
\left(\begin{array}{c}
X \\
Y \\
Z \\
1
\end{array}\right) \mapsto\left(\begin{array}{c}
x \\
y \\
z
\end{array}\right)=\left(\begin{array}{c}
f X \\
f Y \\
Z
\end{array}\right)=\left[\begin{array}{l}
f \\
f
\end{array}\right.
$$

$$
\left.\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]\left(\begin{array}{c}
\mathrm{X} \\
\mathrm{Y} \\
\mathrm{Z} \\
1
\end{array}\right)
$$

II

## Central Projection



$\left.\begin{array}{lll}1 & & 0 \\ & 1 & 0\end{array}\right]\left(\begin{array}{c}\mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \\ 1\end{array}\right)$

$$
\begin{array}{r}
\left(\begin{array}{c}
\mathrm{fX} \\
\mathrm{fY} \\
\mathrm{Z}
\end{array}\right)=\left[\begin{array}{lll}
\mathrm{f} & & \\
& \mathrm{f} & \\
& & 1
\end{array}\right]\left[\begin{array}{l}
1 \\
\\
\\
\mathbf{x}
\end{array}=\mathrm{PX}\right.
\end{array}
$$

$$
\mathrm{P}=\operatorname{diag}(\mathrm{f}, \mathrm{f}, 1)[\mathrm{I} \mid 0]
$$

$\square$

## Principal Point Offset


$(\mathrm{X}, \mathrm{Y}, \mathrm{Z})^{\mathrm{T}} \mapsto\left(\mathrm{fX} / \mathrm{Z}+\mathrm{p}_{\mathrm{x}}, \mathrm{fY} / \mathrm{Z}+\mathrm{p}_{\mathrm{y}}\right)^{\mathrm{T}}$
where $\left(p_{x}, p_{y}\right)^{T}$
are the coordinates of the principal point

## Principal Point Offset

$$
\left.\begin{array}{rl}
\left(\begin{array}{c}
f X+Z p_{x} \\
f Y+Z p_{x} \\
Z
\end{array}\right) & =\left[\begin{array}{lcc}
f & & p_{x} \\
& f & p_{y}
\end{array} 0\right. \\
& \\
& 1
\end{array}\right]\left[\left(\begin{array}{c}
X \\
Y \\
Z \\
1
\end{array}\right) .\right.
$$


where $\quad K=\left[\begin{array}{llc}f & & p_{x} \\ & f & p_{y} \\ & & 1\end{array}\right] \quad \begin{aligned} & \text { is called camera calibration } \\ & \text { matrix }\end{aligned}$

## Camera Rotation and Translation

inhomogeneous coordinates

$$
\begin{array}{r}
\widetilde{\mathbf{X}}_{\mathrm{cam}}=\mathrm{R}(\widetilde{\mathbf{X}}-\widetilde{\mathrm{C}}) \\
\text { where } \quad \widetilde{\mathbf{X}}=\left(\begin{array}{c}
\mathrm{X} \\
\mathrm{Y} \\
\mathrm{Z}
\end{array}\right) \\
\widetilde{\mathbf{X}}_{\mathrm{cam}}=\left(\begin{array}{c}
\mathrm{x}_{\mathrm{cam}} \\
\mathrm{y}_{\mathrm{cam}} \\
\mathrm{z}_{\mathrm{cam}}
\end{array}\right) \\
\widetilde{C}
\end{array}
$$

 world coordinates
represents the same point in camera coordinates
represents the coordinates of the camera origin in the world coordinate frame

Camera Rotation and Translation
homogeneous coordinates
$\mathbf{X}, \mathbf{X}_{c a m} \in \mathrm{R}^{4}$
$\mathbf{X}_{\mathrm{cam}}=\left[\begin{array}{cc}\mathrm{R} & -\mathrm{R} \widetilde{\mathrm{C}} \\ 0 & 1\end{array}\right]\left(\begin{array}{l}\mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \\ 1\end{array}\right)=\left[\begin{array}{cc}\mathrm{R} & -\mathrm{R} \widetilde{\mathrm{C}} \\ 0 & 1\end{array}\right] \mathbf{X}$

$\mathbf{x}=\mathrm{K}[\mathrm{I} \mid 0] \mathbf{X}_{\mathrm{cam}}$ projection to image plane from camera coordinates
$\mathbf{x}=\mathrm{KR}\lfloor\mathrm{I} \mid-\widetilde{\mathrm{C}}] \mathbf{X}$ projection to image plane from world coordinates
$\mathrm{x}=\mathrm{PX}$

$$
\begin{aligned}
& \text { Extrinsic and Intrinsic Parameters }
\end{aligned}
$$

$\mathrm{P}=\mathrm{KR}\lfloor\mathrm{I} \mid-\widetilde{\mathrm{C}}\rfloor \quad \begin{aligned} & \text { projection matrix of a general } \\ & \text { pinhole camera with } 9 \text { DOF }\end{aligned}$
$\mathrm{P}=\mathrm{KR}\lfloor\mathrm{I} \mid-\widetilde{\mathrm{C}}\rfloor \quad \begin{aligned} & \text { projection matrix of a genera } \\ & \text { pinhole camera with } 9 \text { DOF }\end{aligned}$
$x=P X$
where
$K=\left[\begin{array}{llc}f & & p_{x} \\ & f & \\ & p_{y} \\ & & 1\end{array}\right] \quad$ intrinsic camera parameters with 3 DOF
$R, \widetilde{C}$ extrinsic camera parameters with each 3 DOF
(camera orientation and position in world coordinates)

## Camera Rotation and Translation

no explicit
camera center


$$
\begin{aligned}
\widetilde{\mathbf{X}}_{c a m} & =R \widetilde{\mathbf{X}}+t \\
\mathrm{P} & =\mathrm{K}[\mathrm{R} \mid \mathrm{t}] \\
& =\mathrm{K}[\mathrm{R} \mid-\mathrm{R} \widetilde{\mathrm{C}}]
\end{aligned}
$$


where

$$
t=-R \widetilde{C} \quad \text { from } \quad x=K R\lfloor I \mid-\widetilde{C}\rfloor X
$$

## CCD Cameras :: Non-Square Pixel

$\mathrm{m}_{\mathrm{x}}, \mathrm{m}_{\mathrm{y}}$ number of pixels per unit distance

$$
\begin{aligned}
& K=\left[\begin{array}{lll}
m_{x} & & \\
& m_{y} & \\
& & 1
\end{array}\right]\left[\begin{array}{llc}
f & & p_{x} \\
& f & p_{y} \\
& & 1
\end{array}\right] \\
& K=\left[\begin{array}{llc}
\alpha_{x} & & p_{x} \\
& \alpha_{y} & p_{y} \\
& & 1
\end{array}\right] \\
& \mathrm{P}=\mathrm{KR}\lfloor\mathrm{I} \mid-\widetilde{\mathrm{C}}]
\end{aligned}
$$



## Skew Parameter

S skew parameter

$$
K=\left[\begin{array}{ccc}
\alpha_{x} & s & p_{x} \\
& \alpha_{y} & p_{y} \\
& & 1
\end{array}\right] \quad 5 \mathrm{DOF}
$$

$$
\mathrm{P}=\mathrm{KR}\lfloor\mathrm{I} \mid-\widetilde{\mathrm{C}}\rfloor
$$

finite projective camera with 11 DOF

## Finite Projective Camera :: Summary

$$
\begin{aligned}
& \mathbf{x}=P \mathbf{X} \\
& P=\left[\begin{array}{ccc}
\alpha_{x} & S & p_{x} \\
& \alpha_{y} & p_{y} \\
& & 1
\end{array}\right]\left[\begin{array}{cc}
R & -R \widetilde{C} \\
0 & 1
\end{array}\right] \quad \begin{array}{l}
\text { projection matrix } \\
11 \text { DOF (5+3+3) }
\end{array} \\
&=\underbrace{K R[I \mid-\widetilde{C}]}_{\text {non }} \\
& \text { nonsingular }
\end{aligned}
$$

Finite Projective Camera ：：Decomposition of P

$$
\begin{aligned}
\mathrm{M} & =\mathrm{KR} \\
\mathrm{P} & =\mathrm{KR}[\mathrm{I} \mid-\widetilde{\mathrm{C}}] \\
\mathrm{P} & =\mathrm{M}[\mathrm{I} \mid-\widetilde{\mathrm{C}}]
\end{aligned}
$$

decompose projection matrix P in $\mathrm{K}, \mathrm{R}, \mathrm{C}$

$$
\mathrm{P}=\left[\mathrm{M} \mid \mathbf{p}_{4}\right] \quad[\mathrm{K}, \mathrm{R}]=\underbrace{R Q}_{\begin{array}{c}
\mathrm{RQ} \text { matrix } \\
\text { decomposition }
\end{array}}(\mathrm{M}) \quad \widetilde{\mathrm{C}}=-\mathrm{M}^{-1} \mathbf{p}_{4}
$$

## Finite Projective Camera ：：Summary

$$
\begin{aligned}
\mathrm{P} & =\mathrm{KR}[\mathrm{I} \mid-\widetilde{\mathrm{C}}] \\
& =\mathrm{M}[\mathrm{I} \mid-\widetilde{\mathrm{C}}]
\end{aligned}
$$

－camera matrix $P$ are identical with the set of homogeneous $3 \times 4$ matrix for which the left hand $3 \times 3$ submatrix is non singular
－$\quad\{$ finite cameras $\}=\{P \mid$ det $M \neq 0\}=\{P \mid \operatorname{rank}(M)=3\}$
－If $\operatorname{rank}(P)=3$ ，but $\operatorname{rank}(M)<3$ ，then camera at infinity
－if rank（ P$)<3$ the matrix mapping will be a line or a point and not a plane（not a 2D image）

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## Camera Anatomy

- camera center
- column vectors
- principal plane
- axis plane
- principal point
- principal ray



## Camera Center

$$
P C=0 \quad P \text { has a 1D null-space }
$$

we will proof that the 4 -vector $C$ is the camera center
$X(\lambda)=\lambda A+(1-\lambda) C$ points on a line through $A$ and $C$
$\mathrm{X}=\mathrm{PX}=\lambda \mathrm{PA}+(1-\lambda) \mathrm{PC}=\lambda \mathrm{PA} \quad$ since $\quad \mathrm{PC}=0$
all 3D points on the line are mapped on the same 2D image point, and thus the line is a ray through the camera center

Finite cameras:

$$
\begin{aligned}
& C=\binom{\widetilde{C}}{1}=\binom{-\mathrm{M}^{-1} \mathbf{p}_{4}}{1} \\
& C=\binom{\mathbf{d}}{0}, \mathrm{Md}=0
\end{aligned}
$$

Infinite cameras:

## Column Vectors

$\mathbf{p}_{i} \in \mathrm{R}^{3}, i=1, \ldots, 4$

column vectors are the image points, which project the axis directions $(X, Y, Z)$ and the origin
example for the image of the $y$-axis (vanishing pt)

$\mathbf{p}_{4}$ is the image of the world origin

## Row Vectors

$$
\mathbf{p}^{i} \in \mathrm{R}^{4}, i=1, \ldots, 3
$$

represent geometrically particular world planes

$$
\begin{aligned}
\mathbf{P} & =\left[\begin{array}{llll}
p_{11} & p_{12} & p_{13} & p_{14} \\
p_{21} & p_{22} & p_{23} & p_{24} \\
p_{31} & p_{32} & p_{33} & p_{34}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{p}^{1 T} \\
\mathbf{p}^{2 T} \\
\mathbf{p}^{3 T}
\end{array}\right] \text { row vectors } \\
& =\left[\begin{array}{llll}
\mathbf{p}_{1} & \mathbf{p}_{2} & \mathbf{p}_{3} & \mathbf{p}_{4}
\end{array}\right] \quad \text { column vectors }
\end{aligned}
$$

## Row Vectors of the Projection Matrix

$P^{1}$ is defined by the camera center and the line $x=0$ on the image
$\mathrm{P}^{2}$ is defined by the camera center and the line $y=0$ on the image

Example $\mathrm{P}^{2}$

$$
\begin{aligned}
& {\left[\begin{array}{c}
x \\
0 \\
w
\end{array}\right]=\left[\begin{array}{l}
\mathbf{p}^{1^{\top}} \\
\mathbf{p}^{2^{\top}} \\
\mathbf{p}^{3^{\top}}
\end{array}\right]\left[\begin{array}{c}
X \\
Y \\
Z \\
1
\end{array}\right]} \\
& \text { respectively for } P^{1}
\end{aligned}
$$

$$
\mathbf{p}^{2^{T}} \mathbf{X}=0
$$

$$
\mathbf{p}^{2^{\mathrm{T}}} C=0
$$

## Principal Plane

plane through camera center and parallel to the image plane

points $\mathbf{X}$ are imaged on the line at infinity
if $X$ is on the principle plane

$$
\mathbf{p}^{3 T} \mathbf{X}=0
$$

especially
$\mathbf{p}^{3 T} \mathrm{C}=0$

## Principle Point

line through camera center and perpendicular to principle plane is the principle axis
intersection of the principle axis with the image plane is the principle point

$\hat{\mathbf{P}}^{3}=\left(\begin{array}{llll}p_{31} & p_{32} & p_{33} & 0\end{array}\right)^{\mathrm{T}} \quad$ normal direction to principal plane
principle point

$$
\begin{array}{rll}
\mathbf{x}_{0}=\mathbf{P} \hat{\mathbf{P}}^{3}=\mathbf{M} \mathbf{m}^{3} & \text { where } & \mathrm{P}=\left[\mathrm{M} \mid \mathbf{p}_{4}\right] \\
& \text { and } & \mathbf{m}^{3^{\mathrm{T}}} \text { third row of } \mathrm{M}
\end{array}
$$

## Principle Axis Vector

ambiguity that principle axis points towards the front of the camera (positive direction)

towards the front of the camera
$\mathrm{P}_{\mathrm{cam}} \mapsto k \mathrm{P}_{\mathrm{cam}} \quad \mathrm{V} \mapsto k^{4} \mathrm{~V}$
direction unaffected by scaling

$$
\begin{aligned}
& \mathrm{P}=k \mathrm{KR}\lfloor\mathrm{I} \mid-\widetilde{\mathrm{C}}\rfloor=\left[\mathrm{M} \mid \mathrm{p}_{4}\right] \quad \text { since } \quad \operatorname{det}(\mathrm{R})>0 \\
& \mathrm{~V}=\operatorname{det}(\mathrm{M}) \mathrm{m}^{3}
\end{aligned}
$$

## Forward Projection

maps a point in space on the image plane

$$
\mathrm{P}^{3} \rightarrow \mathrm{P}^{2}
$$

$$
\mathrm{x}=\mathrm{PX}
$$

Points at infinity

$$
\mathrm{D}=\left(\mathrm{d}^{\mathrm{T}}, 0\right)^{\mathrm{T}}
$$

$$
\mathrm{x}=\mathrm{PD}=\left[\mathrm{M} \mid \mathrm{p}_{4}\right] \mathrm{D}=\mathrm{Md}
$$

only M effects the projection of points at infinity (vanishing points)

## Back-projection to Rays

points on the reconstructed ray

$$
\begin{aligned}
& \mathrm{PC}=0 \\
& \mathrm{X}=\mathrm{P}^{+} \mathrm{X} \quad \begin{array}{l}
\mathrm{P}^{+}=\mathrm{P}^{\top}\left(\mathrm{PP}^{\top}\right)^{-1} \\
\text { (pseudo-inverse) }
\end{array} \quad \mathrm{PP}^{+}=\mathrm{I}
\end{aligned}
$$

ray is the line formed by those two points

$$
\begin{aligned}
& \mathrm{X}(\lambda)=\mathrm{P}^{+} \mathrm{x}+\lambda \mathrm{C} \\
& \mathrm{D}=\left(\left(\mathrm{M}^{-1} \mathrm{x}\right)^{\mathrm{T}}, 0\right)^{\mathrm{T}} \quad \begin{array}{l}
\text { intersection of the ray with the } \\
\text { plane at infinity }
\end{array} \\
& \mathrm{X}(\mu)=\mu\binom{\mathrm{M}^{-1} \mathrm{x}}{0}+\underbrace{\binom{-\mathrm{M}^{-1} \mathrm{p}_{4}}{1}}_{C}=\binom{\mathrm{M}^{-1}\left(\mu \mathrm{x}-\mathrm{p}_{4}\right)}{1}
\end{aligned}
$$

## Depth of Points

$$
\begin{aligned}
& \mathrm{X}=(X, Y, Z, 1) \\
& \mathrm{C}=(\widetilde{\mathrm{C}}, 1)
\end{aligned}
$$



$$
w=\mathrm{P}^{3^{\mathrm{T}}} \mathrm{X}=\mathrm{P}^{3^{\mathrm{T}}}(\mathrm{X}-\mathrm{C})=\mathrm{m}^{3^{\mathrm{T}}}(\widetilde{\mathrm{X}}-\widetilde{\mathrm{C}})
$$

$$
(\mathrm{PC}=0) \quad(\operatorname{dot} \text { product })
$$

If $\operatorname{det} M>0 ;\left\|\mathrm{m}^{3}\right\|=1$,
then $\mathrm{m}^{3}$ unit vector pointing in positive axis direction

Suppose $P(X, Y, Z, T)^{\mathrm{T}}=w(x, y, 1)^{\mathrm{T}}$. Then

$$
\operatorname{depth}(\mathrm{X} ; \mathrm{P})=\frac{\operatorname{sign}(\operatorname{det} \mathrm{M}) w}{T\left\|\mathrm{~m}^{3}\right\|}
$$

## Depth of Points：examples



$$
\operatorname{depth}(\mathrm{X} ; \mathrm{P})=\frac{\operatorname{sign}(\operatorname{det} \mathrm{M}) w}{\left\|\mathrm{~m}^{3}\right\|} \quad P(X, Y, Z, 1)^{\mathrm{T}}=w(x, y, 1)^{\mathrm{T}}
$$

$$
\mathbf{x}=\mathrm{PX}
$$

$$
\mathbf{x}=\mathrm{PX}
$$

$$
\mathrm{P}=[\mathrm{I} \mid 0]
$$

$$
\mathrm{P}=\mathrm{R}[\mathrm{I} \mid-\widetilde{\mathrm{C}}]
$$

$$
\operatorname{depth}(X ; P)=Z
$$

$$
\operatorname{depth}(\mathrm{X} ; \mathrm{P})=\mathrm{R}^{3^{T}}(\widetilde{X}-\widetilde{\mathrm{C}})
$$

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## Camera Matrix Decomposition

Finding the camera center C

$$
\mathrm{PC}=0
$$

numerically: find right null-space by SVD of $P$
$\mathrm{P}^{3 \times 4}=\mathrm{U}^{3 \times 4} \mathrm{D}^{4 \times 4} \mathrm{~V}^{4 \times 4^{T}} \rightarrow \mathrm{P}^{3 \times 4} \mathrm{~V}^{4 \times 4}=\mathrm{U}^{3 \mathrm{x} 4} \mathrm{D}^{4 \times 4} \rightarrow \mathrm{PV}^{4}=0$

Algebraically:

$$
\begin{array}{ll}
X=\operatorname{det}\left(\left[\mathrm{p}_{2}, \mathrm{p}_{3}, \mathrm{p}_{4}\right]\right) & Y=-\operatorname{det}\left(\left[\mathrm{p}_{1}, \mathrm{p}_{3}, \mathrm{p}_{4}\right]\right) \\
Z=\operatorname{det}\left(\left[\mathrm{p}_{1}, \mathrm{p}_{2}, \mathrm{p}_{4}\right]\right) & T=-\operatorname{det}\left(\left[\mathrm{p}_{1}, \mathrm{p}_{2}, \mathrm{p}_{3}\right]\right)
\end{array}
$$

where $\quad \mathrm{C}=(X, Y, Z, T)^{\mathrm{T}}$

## Camera Matrix Decomposition

Finding the camera center C

$$
\mathrm{PC}=0
$$

Any plane $\pi$ going through $C$ will be a linear combination of the three planes defined by the rows of $P$. Therefore:

$$
\begin{aligned}
\operatorname{det}\left(\| \pi, \mathrm{P}^{\mathrm{T}}\right\rfloor & =0 \\
& =\pi_{1} \operatorname{det}\left(P_{234}\right)-\pi_{2} \operatorname{det}\left(P_{134}\right)+\pi_{3} \operatorname{det}\left(P_{124}\right)-\pi_{4} \operatorname{det}\left(P_{123}\right) \\
X & =\operatorname{det}\left(\left[\mathrm{p}_{2}, \mathrm{p}_{3}, \mathrm{p}_{4}\right]\right) \quad Y=-\operatorname{det}\left(\left[\mathrm{p}_{1}, \mathrm{p}_{3}, \mathrm{p}_{4}\right]\right) \\
Z & =\operatorname{det}\left(\left[\mathrm{p}_{1}, \mathrm{p}_{2}, \mathrm{p}_{4}\right]\right) \quad T=-\operatorname{det}\left(\left[\mathrm{p}_{1}, \mathrm{p}_{2}, \mathrm{p}_{3}\right]\right) \\
& \text { where } \quad \mathrm{C}=(X, Y, Z, T)^{\mathrm{T}}
\end{aligned}
$$

## Camera Matrix Decomposition

Finding the camera orientation and internal parameters

$$
P=\lfloor M \mid-M \widetilde{C}\rfloor=K\lfloor R \mid-R \widetilde{C}\rfloor
$$

$$
\mathrm{K}=\left[\begin{array}{ccc}
\alpha_{x} & s & y_{0} \\
0 & \alpha_{y} & x_{0} \\
0 & 0 & 1
\end{array}\right]
$$

Decompose $\mathrm{M}=\mathrm{KR}$ using RQ decomposition

$$
\square=(Q \nabla)^{-1}=R^{-1} Q^{-1}
$$

Ambiguity removed by enforcing positive diagonal entries

## When is Skew Non－zero？

$$
K=\left[\begin{array}{ccc}
\alpha_{x} & s & p_{x} \\
& \alpha_{y} & p_{y} \\
& & 1
\end{array}\right]
$$


for CCD／CMOS，always s＝0

Image from image， $\mathrm{s} \neq 0$ possible （non coinciding principal axis）
resulting camera：HP
where H is a $3 \times 3$ homography

## Euclidean vs．Projective Spaces

general projective interpretation

$$
P=[3 \times 3 \text { homography }]\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right][4 \times 4 \text { homography }]
$$

Meaningful decomposition in $\mathrm{K}, \mathrm{R}, \mathrm{t}$ requires
Euclidean image and space

Camera center is still valid in projective space
Principal plane requires affine image and space
Principal ray requires affine image and Euclidean space

## Literature on Camera Models

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