

POSE ESTIMATION IN MEDICAL TECHNOLOGY

The seminar I held on 15. June 2004 was divided into three main parts. Under the first title "PE Concept" I tried to give the audiences an overview on the topic which was then intensified by the theoretical part that was deepened by a huge number of equations representing the nature of statistical calculations. After showing the examples and application fields in medicine, the discussion was concluded by a few words. At the end bibliography and my reading suggestions are available.

PE CONCEPT

POSE ESTIMATION is abbreviation of "position and orientation estimation" which is an image registration algorithm classified as "Landmark-Based Registration".

More on image registration algorithms...

Principally algorithms can be observed as Hajnal et. al. stated under two capitals as follows:

Rigid Algorithms:

Corresponding Landmark-Based Registration

Surface-Based Registration

Registration Based on Voxel Intensities

Nonrigid Algorithms

Intersubject Registration

Intrasubject Registration

I constrained my theoretical representation with the first of the rigid algorithms but the next two capitals will be mentioned by my colleagues on the 22&29. June. Nonrigid algorithms, still an area of active research, is out of interest of whole seminar series.

The process of image registration involves finding transformations that relate spatial information conveyed in one image to those in another or in physical space. Shortly it's a matter of transformations, what we will do is to find some parameters to facilitate the lives of surgeons.

To remind ...

What's rigid, what's nonrigid?

A rigid body changes position and orientation without changing size or shape

A nonrigid body deforms by changing shape and size. (Tissues in the humanbody are nonrigid! But using appropriate optimization they can be modelled as if they were rigid, this didn't cause any kind of confusion in the operating rooms so the assumptions are on the right way.) Some possible reasons for changing size and shape are hydratation, systemic blood pressure etc.

THEORETICAL PART

Least-Squares Estimation of Transformation Parameters Between Two Point Patterns

by S. Umeyama

published in the middle of 90's was the heart of my mathematical explanations. To overcome the complexity and to understand what I will talk about find pieces of papers and a pen... Now let's start with stating a lemma, the next step will be to prove the lemma. Thereafter comes the theorem followed by its proof. Never ever believe me, always demand the evidence, which I will give in an excessive amount.

Before diving into theorems, let me summarize the procedure one more time...

We have two sets of points $\{x_i\}$ and $\{y_i\}$; $i=1,2,\dots,n$ in m -dimensional space (m is usually 2 or 3)

We want to find the similarity transformation parameters (R:Rotation; t:Translation; c:Scaling)

- The algorithm for calculating the transformation involves first computing the average or "centroid" of each set of points.
- The difference between the centroids in 3D tells us the translation that must be applied to one set of points.
- This point set is then rotated about its new centroid until the sum of the squared distances between each corresponding point pair is minimized.
- The square root of the mean of this squared distance is often recorded by the registration algorithm.

It is also referred to as the root mean square (RMS) error, residual error. Also, known as the Procrustes problem...

Meanwhile I won't be able to omit the anecdote on Procrustes.

Procrustes was a robber in Greek mythology. He would offer travelers hospitality in his roadside house and the opportunity to stay the night in a bed that would perfectly fit each visitor. As visitors discovered to their cost, however, it was the guest who was altered to fit the bed, rather than the bed to fit the guest. Short visitors were stretched to fit, and tall visitors had suitable parts cut off so they would fit. The result, it seems, was invariably fatal. The hero Theseus put a stop to this unpleasant practice by subjecting Procrustes to his own method.

OK! Back to work...

Achieving our goal can be realized by finding the minimum value of the mean squared error $e^2(R,t,c)$ of these two points. Here is the lemma...

$$e^2(R,t,c) = \frac{1}{n} \sum_{i=1}^n \|y_i - (cRx_i + t)\|^2 \quad \dots(1)$$

Solution of this problem is based on the singular value decomposition of a covariance matrix of the data.

- A and B : mxn matrices
- R: mxm rotation matrix
- UDV^T : singular value decomposition of AB^T

Then the minimum value of $\|A-RB\|^2$ with respect to R is

$$\min_R \|A-RB\|^2 = \|A\|^2 + \|B\|^2 - 2\text{tr}(DS) \quad \dots(2)$$

$$S = \begin{cases} I & \text{if } \det(AB^T) \geq 0 \\ \text{diag}(1,1,\dots,1,-1) & \text{if } \det(AB^T) < 0 \end{cases} \quad \dots(3)$$

$$R = USV^T \quad \dots(4)$$

$$S = \begin{cases} I & \text{if } \det(U)\det(V) = 1 \\ \text{diag}(1,1,\dots,1,-1) & \text{if } \det(U)\det(V) = -1 \end{cases} \quad \dots(5)$$

I would like to attract your attention to the product of the two determinants above. If its value is 1, it means that the image is not rotation symmetric, this is also what we prefer. If the value equals to -1, **DANGER**, image is rotation symmetric, the surgeon can operate the wrong side of the brain.

Let's prove it...In fact it's easier to perceive if the proof is explained in two sections.

1.Half

We are defining the function below

- Objective funktion F

$$F = \|A - RB\|^2 + \text{tr}(L(R^T R - I)) + g\{\det(R) - 1\} \dots(6)$$

g: Lagrange multiplier

L: symmetric matrix of Lagrange multipliers

$$\frac{\partial F}{\partial R} = -2AB^T + 2RBB^T + 2RL + gR = 0 \dots(7)$$

$$\frac{\partial F}{\partial L} = R^T R - I = 0 \dots(8)$$

$$\frac{\partial F}{\partial g} = \det(R) - 1 = 0 \dots(9)$$

R: rotation matrix

adj(R^T): adjoint matrix of R^T

$$\frac{\partial}{\partial R} \det(R) = \text{adj}(R^T) = \det(R^T) (R^T)^{-1} = R \dots(10)$$

$$RL' = AB^T \quad L' = BB^T + L + \frac{1}{2}gI \dots(11)$$

$$L'R^T = BA^T \dots(12)$$

$$(L')^2 = BA^T AB^T = VD^2V^T \dots(13)$$

$$L' = VDSV^T \dots(14)$$

$S = \text{diag}(s_i), \quad s_i = 1, \text{ or } -1$

$$\begin{aligned} \det(L') &= \det(VDSV^T) \\ &= \det(V)\det(D)\det(S)\det(V^T) \\ &= \det(D)\det(S) \end{aligned} \quad \dots(15)$$

$$\begin{aligned} \det(L') &= \det(R^T AB^T) \\ &= \det(R^T)\det(AB^T) \\ &= \det(AB^T) \end{aligned} \quad \dots(16)$$

From the equality of (15)&(16)

$$\det(D)\det(S) = \det(AB^T) \quad \dots(17)$$

- Singular matrices are nonnegative, (!Attention!)
 $\det(D) = d_1 d_2 \dots d_m \geq 0$

$$\begin{aligned} \det(S) &= 1 && \text{if } \det(AB^T) > 0 \\ \det(S) &= -1 && \text{if } \det(AB^T) < 0 \end{aligned}$$

$$\begin{aligned} \|A - RB\|^2 &= \|A\|^2 + \|B\|^2 - 2\text{tr}(AB^T R^T) \\ &= \|A\|^2 + \|B\|^2 - 2\text{tr}(R^T AB^T) \\ &= \|A\|^2 + \|B\|^2 - 2\text{tr}(L') \end{aligned} \quad \dots(18)$$

Substituting (14) into (18), we have

$$\begin{aligned} \|A - RB\|^2 &= \|A\|^2 + \|B\|^2 - 2\text{tr}(VDSV^T) \\ &= \|A\|^2 + \|B\|^2 - 2\text{tr}(DS) \\ &= \|A\|^2 + \|B\|^2 - 2(d_1 s_1 + d_2 s_2 + \dots + d_m s_m) \end{aligned} \quad \dots(19)$$

The minimum value of $\|A-RB\|^2$ is achieved when $s_1=s_2=\dots=s_{m-1}=1$ if $\det(AB^T) < 0$. 1. half is over. Now 2. half of the proof begins.

- We determine a rotation matrix R achieving the previous minimum value.

$$\text{rank}(AB^T)=m \quad \text{then } L' \text{ is nonsingular} \\ \text{thus it has its inverse}$$

Just to remind

$$(L')^{-1} = (VDSV^T)^{-1} = VS^{-1}D^{-1}V^T \\ [S^{-1}=S, SD^{-1}=D^{-1}S]$$

Back to ...(11), now modified form

$$R = AB^T (L')^{-1} = UDV^T VD^{-1}SV^T = USV^T \quad \dots(20)$$

- When $\text{rank}(AB^T)=m-1$ then from (11)&(14)

$$RVDSV^T = UDV^T \quad \dots(21)$$

Multiplying V by both sides from right and using $DS=D$ (since $d_m=0$ and $s_1=s_2=\dots=s_{m-1}=1$)

$$RVD = UD \quad \dots(22)$$

Defining an orthogonal matrix Q as follows:

$$Q = U^T RV \quad \dots(23)$$

$$\text{We have} \quad QD = D \quad \dots(24)$$

Let the column vectors of Q be $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$ ($Q = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m]$).

The following equations are obtained by comparing both sides of (24)

$$d_i \mathbf{q}_i = d_i \mathbf{e}_i \quad , 1 \leq i \leq m-1 \quad \dots(25)$$

$$\mathbf{q}_i = \mathbf{e}_i \quad \dots(26)$$

\mathbf{e}_i is a unit vector which has 1 as an *i*th element

$$\mathbf{e}_i = (0, 0, \dots, 1, \dots, 0)^T \quad \dots(27) \\ \textit{i} \textit{th element}$$

The last column vector q_m of Q is orthogonal to all other vectors q_i ($1 \leq i \leq m-1$) since Q is an orthogonal matrix. Now we have

$$q_m = e_m \quad \text{or} \quad q_m = -e_m \quad \dots(28)$$

$$\det(Q) = \det(U^T) \det(R) \det(V) = \det(U) \det(V) \quad \dots(29)$$

$$\begin{aligned} \det(Q) &= 1 && \text{if } \det(U)\det(V) = 1 \\ \det(Q) &= -1 && \text{if } \det(U)\det(V) = -1 \end{aligned}$$

$$R = UQV^T = USV^T \quad \dots(30)$$

$$S = \begin{cases} I & \text{if } \det(U)\det(V) = 1 \\ \text{diag}(1, 1, \dots, 1, -1) & \text{if } \det(U)\det(V) = -1 \end{cases} \quad \dots(31)$$

2.half is also over. If everything till here is clear enough, we are writing the theorem.

- Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be corresponding patterns in m -dimensional space.
- The minimum value ε^2 of the mean squared error

$$\varepsilon^2(R, t, c) = \frac{1}{n} \sum_{i=1}^n \|y_i - (cRx_i + t)\|^2 \quad \dots(32)$$

of these two patterns with respect to the similarity transformation parameters (R :Rotation, t :translation, and c :scaling) is given as follows:

- Error
$$\varepsilon^2 = \sigma_y^2 - \frac{\text{tr}(DS)^2}{\sigma_x^2} \quad \dots(33)$$

- Mean vector of X
$$\mu_x = \frac{1}{n} \sum_{i=1}^n x_i \quad \dots(34)$$

- Mean vector of Y
$$\mu_y = \frac{1}{n} \sum_{i=1}^n y_i \quad \dots(35)$$

- Variance around the mean vector of X

$$\sigma_x^2 = \frac{1}{n} \sum_{i=1}^n \|x_i - \mu_x\|^2 \quad \dots(36)$$

- Variance around the mean vector of Y

$$\sigma_y^2 = \frac{1}{n} \sum_{i=1}^n \|y_i - \mu_y\|^2 \quad \dots(37)$$

- Covariance matrix of X and Y

$$\Sigma_{xy} = \frac{1}{n} \sum_{i=1}^n (y_i - \mu_y)(x_i - \mu_x)^T \quad \dots(38)$$

When $\text{rank}(\Sigma_{xy}) \geq m-1$, the optimum transformation parameters are determined uniquely as follows:

$$R = USV^T \quad \dots(40)$$

$$t = \mu_y - cR\mu_x \quad \dots(41)$$

$$c = \frac{1}{\sigma_x^2} \text{tr}(DS) \quad \dots(42)$$

Just to remind...

$$S = \begin{cases} I & \text{if } \det(U)\det(V)=1 \\ \text{diag}(1,1,\dots,1,-1) & \text{if } \det(U)\det(V)=-1 \end{cases} \quad \dots(43)$$

$$S = \begin{cases} I & \text{if } \det(\Sigma_{xy})=0 \\ \text{diag}(1,1,\dots,1,-1) & \text{if } \det(\Sigma_{xy}) \neq 0 \end{cases} \quad \dots(39)$$

Proof of theorem can be made as follows:

- X & Y are $m \times n$ matrices

$X = [x_1, x_2, \dots, x_n]$ and $Y = [y_1, y_2, \dots, y_n]$, then $e^2(R, t, c)$ is reformulated as follows:

$$e^2(R, t, c) = \frac{1}{n} \|Y - cRX - th^T\|^2 \quad \dots(44)$$

Where $h=(1,1,\dots,1)^T$... (45)

K: $n \times n$ normalisation matrix ($K^2=K^T=K$)

$$K = I - \frac{1}{n}hh^T$$

■ (36)-(38) rewritten:

$$\sigma_x^2 = \frac{1}{n} \|XK\|^2 \quad \dots(46)$$

$$\sigma_y^2 = \frac{1}{n} \|YK\|^2 \quad \dots(47)$$

$$\sum_{xy} = \frac{1}{n} YKX^T \quad \dots(48)$$

$$X = XK + \frac{1}{n}Xhh^T \quad \dots(49)$$

$$Y = YK + \frac{1}{n}Yhh^T \quad \dots(50)$$

Replacing redefined terms in the error equation:

$$\begin{aligned} e^2(R,t,c) &= \frac{1}{n} \left\| YK + \frac{1}{n}Yhh^T - cRXK - \frac{c}{n}RXhh^T - th^T \right\|^2 \\ &= \frac{1}{n} \left\| YK - cRXK + \left(\frac{1}{n}Yh - \frac{c}{n}RXh - t \right) h^T \right\|^2 \\ &= \frac{1}{n} \left\| YK - cRXK - t'h^T \right\|^2 \\ &= \frac{1}{n} \left\{ \|YK - cRXK\|^2 + \|t'h^T\|^2 - 2\text{tr}(K(Y^T - cX^T R^T)t'h^T) \right\} \end{aligned}$$

... (51)

To facilitate calculation process:

$$t' = -\frac{1}{n}Yh + \frac{c}{n}RXh + t \quad \dots(52)$$

Then

$$\begin{aligned} \text{tr}(K(Y^T - cX^T R^T)Y'h^T) &= \text{tr}(h^T(I - \frac{1}{n}hh^T) \times (Y^T - cX^T R^T)Y') \\ &= \text{tr}((h - h^T) \times (Y^T - cX^T R^T)Y') \\ &= 0 \end{aligned} \quad \dots(53)$$

$$\|t'h^T\|^2 = n\|t'\|^2 \quad \dots(54)$$

Now we have

$$\varepsilon^2(R, t, c) = \frac{1}{n} \|YK - cRXK\|^2 + \|t'\|^2 \quad \dots(55)$$

t' must be equal to 0 to minimize error

$$t = \frac{1}{n}Yh - \frac{c}{n}RXh = \mu_y - cR\mu_x \quad \dots(56)$$

When UDV^T is a singular value decomposition of $\Sigma_{xy} = (1/n)YKX^T$, a singular value decomposition of $YK(cXK)^T = cYKK^T X^T = cYKX^T$ is $cnUDV^T$.

- The minimum value $\varepsilon^2(c)$ of $(1/n)\|YK - cRXK\|^2$ with respect to R is given from the lemma as follows:

$$\begin{aligned} \varepsilon^2(c) &= \frac{1}{n} \left\{ \|YK\|^2 + \|cXK\|^2 - 2\text{tr}(cnDS) \right\} \\ &= \sigma_y^2 + c^2 \sigma_x^2 - 2c\text{tr}(DS) \end{aligned} \quad \dots(57)$$

where

$$S = \begin{cases} I & \text{if } \det(\Sigma_{xy}) \geq 0 \\ \text{diag}(1, 1, \dots, 1, -1) & \text{if } \det(\Sigma_{xy}) < 0 \end{cases} \quad \dots(58)$$

- From the lemma, if $\text{rank}(\Sigma_{xy}) \geq m-1$

$$R=USVT \quad \dots(59)$$

$$S=I \quad \begin{array}{l} \text{if } \det(U)\det(V)=1 \\ \text{diag}(1,1,\dots,1,-1) \quad \text{if } \det(U)\det(V)=-1 \end{array} \quad \dots(60)$$

Finally, since $\varepsilon^2(c)$ is a quadratic form of c , the minimum value of $\varepsilon^2(c)$ is achieved when,

$$c = \frac{\text{tr}(DS)}{\sigma_x^2} \quad \dots(61)$$

And the minimum value ε^2 is

$$\begin{aligned} \varepsilon^2 &= \sigma_y^2 + \left\{ \frac{\text{tr}(DS)}{\sigma_x^2} \right\}^2 \sigma_x^2 - 2 \left\{ \frac{\text{tr}(DS)}{\sigma_x^2} \right\} \text{tr}(DS) \\ &= \sigma_y^2 - 2 \frac{\text{tr}(DS)^2}{\sigma_x^2} \end{aligned} \quad \dots(62)$$

All the parameters, R, t and c are obtained.

A short summary for understanding of the geometrical meaning of the method...

We have two sets of points as seen in Figure1

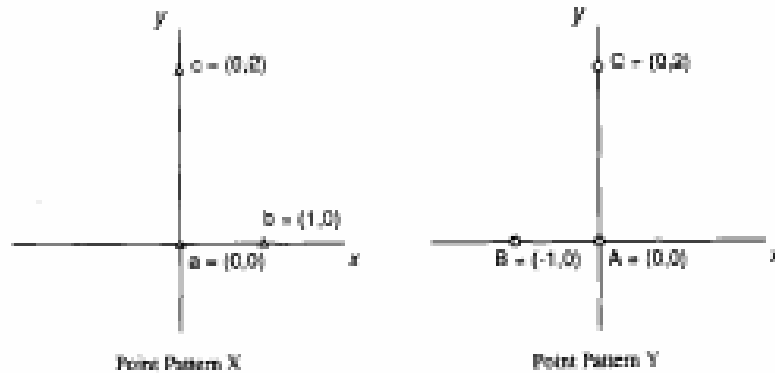


Fig. 1. Two point patterns X and Y consisting of three points in two-dimensional space.

If every factor in surrounding had been ideal, we would get the 100% matching of points as seen in Figure2

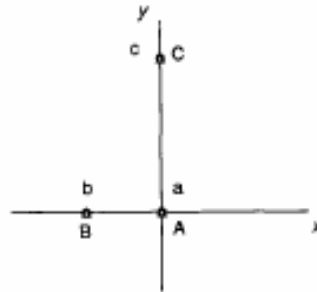


Fig. 2. The matching result without the proper condition of R.

But the result we get is usually like the one in Figure3

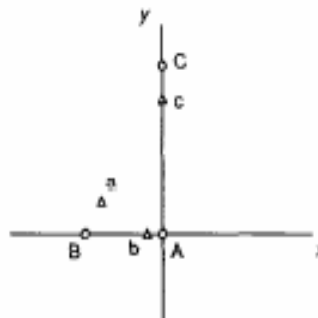


Fig. 3. The matching result with the proper condition of R.

MEDICAL APPLICATIONS

All clinical branches of medicine feel the need for image registration techniques leading to better diagnosis and operation phases during cure processes. To help physicians during those procedures and to accelerate development of surgical operation techniques, the necessity of cooperation of engineers and doctors cannot be rejected.

Stereotactic trusses used by neurosurgeons, preparation of digital anatomy atlases, pose estimation of cylindrical fragments for bone fracture reduction and computer-assisted surgery in shoulder joint replacement made use of orthopaedists and etc. are only a few examples of application probabilities of advanced engineering techniques aiming at the well-being of humankind.

CONCLUSION

The great majority of image registration algorithms was restricted to rigid-body transformations.

Nonrigid registration to compensate for tissue deformation of differences between subjects has become an area of active research.

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